Differential Geometry I $\boxed{\text{Week 6}}$

Last week: We talked about the curvature diagram of a curve in the plane. When $\gamma: I \to \mathbb{R}^2$ is naturally parametrized, it is the graph of the function $s \longrightarrow \kappa^{or}(s)$

- If γ is not naturally parametrized: It is the curve $t \longrightarrow (s(t), \kappa^{or}(t))$.

If γ is a closed C^2 curve: $\kappa^{or}(0) = \kappa^{or}(\ell)$ (ℓ = length of γ)

4 vertex theorem: Let $\gamma: [a,b] \to \mathbb{R}^2$ be a simple closed curve of class C^3. Then γ has at least 4 vertices.

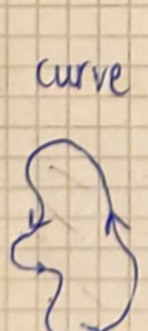
Remarks:
- Ellipse: Exactly 4 vertices: 
- Circle: Every point is a vertex
- If the curve is not simple: Not true

  e.g.  has 2 vertices.

Proof (by Osserman; see also the book by do Carmo for an alternative proof).

Assume without loss of generality that γ is parametrized by arc length. For simplicity, we will denote κ_γ^{or} by κ.

- The curve is closed simple, so divides the plane into two regions:

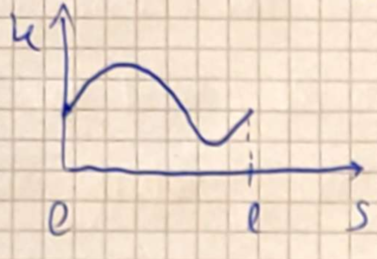


We will assume it is counter-clockwise oriented.

- A circle that is counterclockwise oriented has $\kappa > 0$

• Since γ is C^3 and closed:

$$\kappa(0) = \kappa(l) \quad \text{and} \quad \dot{\kappa}(0) = \dot{\kappa}(l)$$

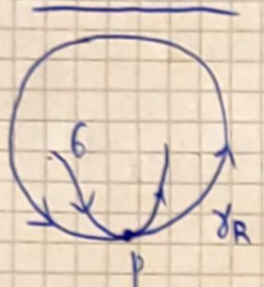


So κ has at least two extrema

– we will show it has at least 4.

We will use two lemmas:

Lemma 1:



Assume that γ_R is a counter-clockwise circle of radius R.

Assume that σ is a curve inside γ_R, such that

$\sigma(t_0) = \gamma_R(t_0)$ & $T_\sigma(t_0) = T_{\gamma_R}(t_0)$.

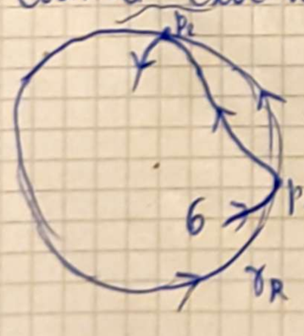
Then $\kappa_\sigma(t_0) \geq \frac{1}{R} = \kappa_{\gamma_R}(t_0)$

If we replace "inside" with "outside": $\kappa_\sigma(t_0) \leq \frac{1}{R}$.

We proved this last time, when we talked about curves which are graphs of functions (I can locally express any curve as a graph of a function, if I identify the x axis with the tangent line).

Lemma 2:

Let σ be a curve inside the circle γ_R (parametrized counter-clockwise) such that

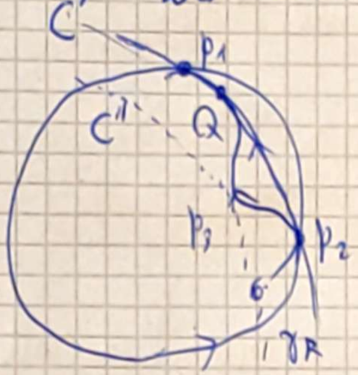


i) $\sigma \cap \gamma_R = \{p_1, p_2\}$, with p_1, p_2 in the same semi-circle of γ_R.

ii) At the two points p_1, p_2: $T_\sigma = T_{\gamma_R}$

Then $\exists$ point p_3 on σ between p_1 and p_2 such that $\kappa_\sigma|_{p_3} < \frac{1}{R}$.

<u>Proof</u>: Choose a point Q on γ between p_1 and p_2 that lies strictly inside γ_R. Then, if Q is sufficiently



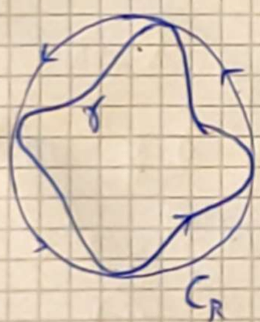
close to p_1: The circle going through $p_1 Q p_2$ is close to γ_R, but has strictly bigger radius $R' > R$. Let's call this circle C'.

Translate C' away from $\{p_1, p_2\}$; there is a final time when it touches $\sigma|_{[p_1, p_2]}$, say at a point p_3.

At this critical time: $\sigma|_{[p_1,p_2]}$ is outside C'' (the translate of C'), touching only at p_3. By the previous lemma:

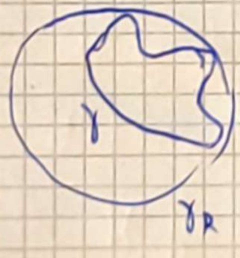
$$k_\sigma \leq \frac{1}{R'} < \frac{1}{R}. \qquad \square$$

<u>Proof of the theorem (continued)</u>:



Let C_R be the circumscribed circle to the curve (i.e. the circle with the smallest radius).

- Not all points of $\gamma \cap C_R$ can lie in the same open semicircle: If that was the case



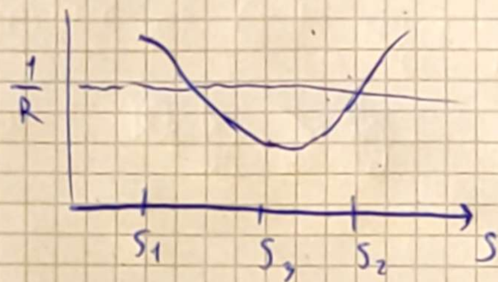
(see the picture on the left) then we could move the circle to the right and shrink it, hence contradicting the minimality of the radius.

Hence, $\gamma \cap C_R$ contains at least two points, and $\gamma \cap C_R$ is not contained inside an open semicircle (so if $\gamma \cap C_R$ only has two points: They have to be antipodal).

Let p_1, p_2 be two points on $\gamma \cap C_R$ such that they belong to the same closed semicircle and such that there exists a point p_3 on γ between p_1, p_2 lying strictly inside the circle.

By the previous lemma: $\kappa|_{p_1}, \kappa|_{p_2} \geq \frac{1}{R}$

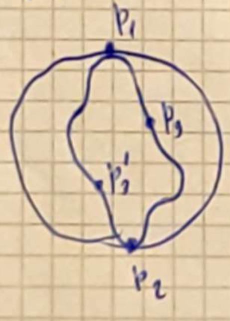
$\kappa|_{p_3} < \frac{1}{R}$



So κ has a local minimum between p_1, p_2.

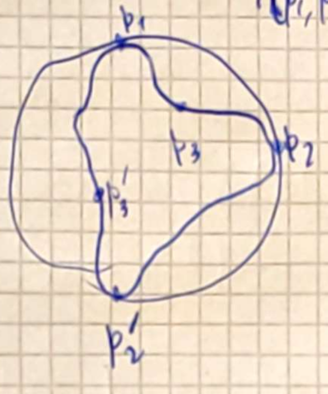
We consider two cases:

- If $\gamma \cap C_R$ is only two points $\{p_1, p_2\}$: They have to be antipodal, so we can ~~conclude that as local maximum p_3 on the other semi circle as well~~ repeat the above process twice (once for each semi-circle)



$\rightsquigarrow$ two local minima p_3, p_3'

- If $\gamma \cap C_R$ has more than two points: I can form at least two pairs $\{p_1, p_2\}$, $\{p_1', p_2'\}$ of touching points with $\sigma|_{(p_1, p_2)} \cap \sigma|_{(p_1', p_2')} = \emptyset$



So I get two distinct local minima p_3, p_3' in the corresponding intervals.

Overall: $\kappa(s)$ has at least two local minima. Since it must have a local maximum as well, it has at least three local extrema. Since $\kappa(0) = \kappa(l)$, this means it must have at least four local extrema.

$\boxtimes$

Note: The above proof in fact gives at least six vertices in the case when $\gamma \cap C_R$ consists of more than two points.

In order to talk about surfaces, submanifolds etc, we have to review some notions from differential calculus.

<u>Directional derivatives and partial derivatives.</u>

Let $U \subseteq \mathbb{R}^m$ be a domain (i.e. open and connected set) and $f: U \to \mathbb{R}^n$. For $p \in U$, $v \in \mathbb{R}^n$, the directional derivative of f in the direction v at the point p is

$$D_v f(p) = \frac{d}{dt} f(p+tv)\Big|_{t=0} = \lim_{t \to 0} \frac{f(p+tv) - f(p)}{t} \in \mathbb{R}^n$$

(if the limit exists)

If $\{e_1, .., e_m\}$ the Cartesian basis of $\mathbb{R}^m$ and $(x^1, .., x^m)$ the associated coordinates:

Partial derivative of f: $\frac{\partial f}{\partial x^i}(p) = D_{e_i} f(p)$.

<u>Note</u>: The existence of all partial derivatives $\frac{\partial f}{\partial x^i}$, $i = 1, .., m$, does not imply the ~~...~~ existence of the directional derivative in every direction: E.g. $f: \mathbb{R}^2 \to \mathbb{R}$,

$$f(x,y) = \begin{cases} \frac{xy}{(x^2+y^2)^{3/4}}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0) \end{cases}$$

has $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$, but for $v \neq e_1, e_2$, $D_v f(0,0)$ is not defined (the limit is $+\infty$).

<u>Def</u>: A function $f: U \to \mathbb{R}^n$ is of class C^1 if it is continuous and $\frac{\partial f}{\partial x^i}$, $i = 1, .., m$, exist on U and are continuous. Similarly, of class C^k if all derivatives of order up to k exist and are continuous.

<u>Notation</u>: $C^k(U, \mathbb{R}^n)$: Functions taking values in $\mathbb{R}^n$.

$$C^k(U) = C^k(U, \mathbb{R})$$

$$C^\infty(U, \mathbb{R}^n) = \bigcap_{k>0} C^k(U, \mathbb{R}^n).$$

<u>Def</u>. If $f: U \to \mathbb{R}^n$, <u>Jacobian matrix</u>:

$$Df = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots \\ \vdots & \vdots & \\ \frac{\partial f_n}{\partial x_1} & & \end{pmatrix}$$

Infinitesimally: Describe the "linearization" of f.

If $n=m$: Jacobian

$$J_f = \det(Df).$$

So $J_f(p) \neq 0 \Rightarrow Df(p)$ is invertible.

Definition:

1. Diffeomorphism of class C^k between open sets U, V: A bijection $f: U \to V$ such that f and f^{-1} are of class C^k

   ($k=0$: Homeomorphism).

   (Theorem: "Invariance of domain": If $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ are open and homeomorphic, then $n=m$)

2. A map $f: U \to \mathbb{R}^n$ is a local diffeomorphism of class C^k at $p \in U$ if $\exists U' \subseteq U$ open containing p such that $V' = f(U')$ is open and $f: U' \to V'$ is a diffeomorphism of class C^k.

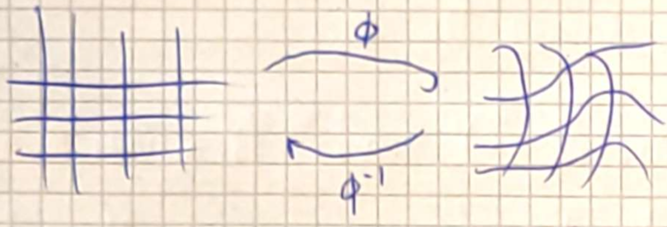
3. $f: U \to \mathbb{R}^n$ is a local diffeomorphism if it is a local diffeomorphism around each $p \in U$.

Remarks:
- If f is a local diffeomorphism: It is not necessarily injective (or surjective)
- If f is a C^1 homeomorphism: It is not necessarily a diffeomorphism of class C^1 (e.g. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$)

  But: If f is C^k and a diffeomorphism of class C^1 $\Rightarrow$ it is a diffeomorphism of class C^k.

Def: A system of (curvilinear) coordinates of class C^k on the open set $U \subseteq \mathbb{R}^n$ is a set $y_1, \dots, y_n : U \to \mathbb{R}$

Such that $\phi : (x_1, \dots, x_n) \longrightarrow (y_1, \dots, y_n)$ is a diffeomorphism of class C^k from U to $V = f(U)$



- Note: i-th coordinate curve: The curve given by

$$\{y^1, \dots, y^{i-1}, y^{i+1}, \dots, y^n = \text{const}\}$$

Differentiable functions in the sense of Fréchet.

Def: $f : U \subseteq \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is differentiable in the sense of Fréchet at $p \in U$ if $\exists$ linear map $A : \mathbb{R}^m \to \mathbb{R}^n$ such that

$$f(x) - f(p) = A \cdot (x - p) + o(\|x - p\|).$$

Note: If A exists, it is unique; we denote $df_p := A$

Alternative expression: If $x = p + h$

$$f(p+h) = f(p) + df_p(h) + o(\|h\|).$$

So $df_p(h) = \lim\limits_{t \to 0} \left(\dfrac{f(p+th) - f(p)}{h} \right) = \dfrac{d}{dt} f(p+th)\Big|_{t=0}$

is the directional derivative of f in the direction h.

Examples: i) If $f : I \to \mathbb{R}$ is differentiable at p, $df_p(h) = f'(p) \cdot h$

ii) If $f : \mathbb{R}^m \to \mathbb{R}^n$ is affine, i.e. $f(x) = Ax + b$,

then $df_p(h) = A \cdot h$

iii) If $f: M_n(\mathbb{R}) \to M_n(\mathbb{R})$ is $f(A) = A^2$, $df_A(H) = A \cdot H + H \cdot A$

... is $f(A) = A^{-1}$: $df_A(H) = -A^{-1} H A^{-1}$

iv) The map $\det: GL_n(\mathbb{R}) \to \mathbb{R}$ has

$$d\det_A(H) = \det A \cdot tr(A^{-1} \cdot H).$$

<u>Proposition (Chain rule)</u>: Let $U \subseteq \mathbb{R}^m$, $V \subseteq \mathbb{R}^n$ be open and $f: U \to V$, $g: V \to \mathbb{R}^s$ be such that f is Fréchet differentiable at p and g is Fréchet differentiable at $f(p)$. Then

$$d(g \circ f)_p = dg_{f(p)} \circ df_p$$

<u>Proof</u>:

Use that $f(p+h) - f(p) = df_p(h) + o(\|h\|)$, $g(q+v) - g(q) = dg_q(v) + o(\|v\|)$

To compute

$$\begin{aligned} g \circ f(p+h) - g \circ f(p) &= g\left(f(p) + df_p(h) + o(\|h\|)\right) - g(f(p)) \\ &= \cancel{g(f(p))} + dg_{f(p)}\left(df_p(h) + o(\|h\|)\right) + o\left(\|df_p(h)\| + o(\|h\|)\right) \\ &\quad \cancel{- g(f(p))} \\ &= dg_{f(p)}\left(df_p(h)\right) + o(\|h\|). \end{aligned}$$

$\square$

<u>Proposition</u>:

If $f: U \subseteq \mathbb{R}^m \to \mathbb{R}^n$ is everywhere Fréchet differentiable and $\|df_p\| \leq C$ $\forall p$, then f is C-Lipschitz, i.e.

$$\forall x, y \in U: \quad \|f(x) - f(y)\| \leq C \cdot \|x - y\|.$$

<u>Proof</u>:

$\forall x, y \in U$: If $\sigma(t) = (1-t)x + ty$, $t \in [0,1]$ is the line segment connecting x, y, then

$$f(x)-f(y) = f(\sigma(0)) - f(\sigma(1)) = -\int_0^1 \frac{d}{dt}\left(f(\sigma(t))\right) dt$$

$$= -\int_0^1 df_{\sigma(t)}(\dot{\sigma}(t))\, dt = -\int_0^1 df_{\sigma(t)}(y-x))\, dt$$

So $\|f(x)-f(y)\| = \|\int_0^1 df_{\sigma(t)}(x-y)\,dt\| \leq \int_0^1 \|df_{\sigma(t)}\| \cdot \|x-y\|\, dt$

$$\leq C\,\|x-y\| \quad \square$$

Theorem: If $f: U \subseteq \mathbb{R}^m \to \mathbb{R}^n$ is C^1, then f is Fréchet differentiable everywhere and, $\forall p \in U$ $\quad df_p = \left(\frac{\partial f_i}{\partial x_j}(p)\right)$. $\leftarrow$ Jacobian matrix.

In particular, if f is of class C^1, then $p \to df_p$ is a continuous map.

Gradient:

Let $(x_1, \dots, x_m)$ be the Cartesian coordinates on $\mathbb{R}^m$. Since each $x_i: \mathbb{R}^m \to \mathbb{R}$ is a linear function:

$$dx_i(v) = v_i \quad \forall v \in \mathbb{R}^m,$$ at every point in U.

If $f: U \to \mathbb{R}$ is Fréchet differentiable at $p \in U$:

$\forall v \in \mathbb{R}^m$: We can decompose $v = \sum_{i=1}^m v_i e_i$

So $df_p(v) = \sum_{i=1}^m v_i\, df_p(e_i)$.

But $df_p(e_i) = \lim_{t\to 0} \frac{f(p+te_i)-f(p)}{t} = \frac{\partial f}{\partial x_i}(p)$

So $df_p(v) = \sum_{i=1}^m \frac{\partial f}{\partial x_i}(p) \cdot v_i = \langle \vec{\nabla} f, v \rangle,$

Another way of expressing it:

$$\boxed{df_p = \sum_{i=1}^{m} \frac{\partial f}{\partial x_i}(p)\, dx^i}$$

(True also when f is vector valued)

Note: For $f: U \to \mathbb{R}$, $p \in U$:

$df_p : \mathbb{R}^m \to \mathbb{R}$ is a covector of $\mathbb{R}^m$,

while $\vec{\nabla} f_p$ is a vector of $\mathbb{R}^m$

So we represent $df_p = \left(\frac{\partial f}{\partial x_1} \ \cdots \ \frac{\partial f}{\partial x_m} \right)$ (row), $\vec{\nabla} f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_m} \end{pmatrix}$ (column)

The inverse function theorem:

Let $U \subseteq \mathbb{R}^n$ be open and $f \in C^k(U, \mathbb{R}^n)$, $k \geq 1$. If $J_f(p) \neq 0$, then f is a local diffeomorphism around p.

(Proof: In the analysis I course).

Corollary: A map $f: U \to V$, $U, V \subseteq \mathbb{R}^n$ open, is a global diffeomorphism if it is bijective and $J_f \neq 0$ everywhere.